

# Lup-Like Cantilever Beam for Small Deflection

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## Abstract

A lup-like cantilever beam are discussed in this work. For small deflection it can be approximated as a spring-mass system with certain spring constant whose effective mass is larger than the usual constant rectangular cross section cantilever beam. A new parameter  $\beta$  is introduced to relates some the properties of lup-like cantilever beam to the usual one. Influence of beam width  $B_0$  and head width  $B_t$  to value of  $\beta$  is also presented.

## 1 Introduction

Cantilever beams play important role in many today applications. It is used as components in common bridge [1], railway bridge [2], and aeroplane wing [3]. In smaller scale it is in sensors for viscosity [4] and acceleration [5]. In nanoscopic scale application for atomic force microscope (AFM) is already common [6], even it can be used to measure weight of single virus [7]. Common form for cantilever beam is with constant rectangular cross section, where different form will have its own first mode natural frequency [8]. A recent application uses also a lup-like form which is not yet common [9], that needs a theoretical approach to characterize the cantilever beam, which is discussed in this work. Limitation to small deflection is still required here.

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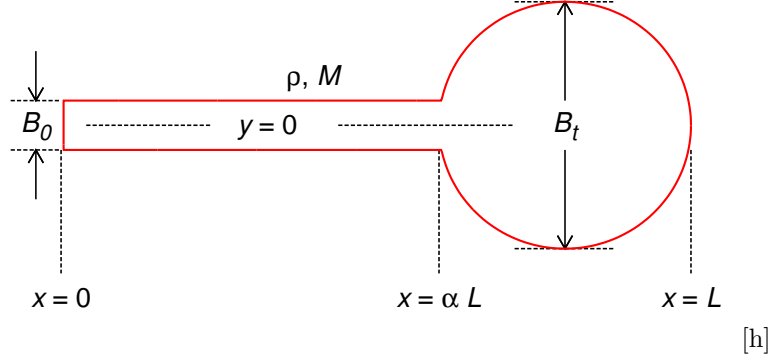


Figure 1: Model of lup-like cantilever beam with mass  $M$ , density  $\rho$ , arm width  $B_0$ , arm length  $\alpha L$ , head width  $B_t$ , head length  $B_t - \delta$ , and thickness  $H$  (in  $z$  direction, perpendicular to drawing plane).

## 2 Mass and area of moment inertia

A cantilever beam that has a lup-like form is illustrated in Figure 1. It has length of  $L$ , thickness  $H$ , density  $\rho$ , and mass  $M$ . The cantilever consists of two parts, which are arm and head. Arm has length of  $\alpha L$  and width of  $B_0$ , while head has length of  $B_t - \delta$  and width of  $B_t$ .

A function to represent width of cantilever beam  $B$  as function of  $x$  for this case can be defined as

$$B(x) = \begin{cases} B_0, & 0 \leq x < \alpha L, \\ 2\sqrt{\frac{1}{4}B_t^2 - \left[x - \left(\alpha L + \frac{1}{2}\sqrt{B_t^2 - B_0^2}\right)\right]^2}, & \alpha L \leq x \leq L, \end{cases} \quad (1)$$

with head length to beam length ratio  $(1 - \alpha)$  defined as

$$(1 - \alpha) = \frac{B_t - \delta}{L}, \quad (2)$$

and parameter  $\delta$  as

$$\delta = \frac{1}{2} \left( B_t - \sqrt{B_t^2 - B_0^2} \right), \quad (3)$$

which makes the cross section of the beam

$$A(x) = \begin{cases} HB_0, & 0 \leq x < \alpha L, \\ 2H\sqrt{\frac{1}{4}B_t^2 - \left[x - \left(\alpha L + \frac{1}{2}\sqrt{B_t^2 - B_0^2}\right)\right]^2}, & \alpha L \leq x \leq L. \end{cases} \quad (4)$$

Mass of the beam with constant density  $\rho$  is then determined using

$$M = \int_0^L \rho A(x) dx, \quad (5)$$

which gives result

$$M = \alpha \rho H B_0 L + \frac{1}{4} (1 - \alpha) \rho H B_t L \left[ \pi - \sin^{-1} \left( \frac{B_0}{B_t} \right) + \frac{B_0 \sqrt{B_t^2 - B_0^2}}{B_t^2} \right]. \quad (6)$$

In Equation (6) parameter  $\alpha$  and  $B_t$  are dependent to each other, e.g. value  $\alpha = 1$  corresponds to  $B_t = 0$ , while  $\alpha = 0$  corresponds to  $B_t = L$  as in Equation (2), and at these limits it is required that  $\delta = 0$ .

Area of moment inertia for a cantilever beam with width  $B$  which is deflected in the direction of its thickness  $H$  is [10]

$$I = \frac{1}{12} H^3 B. \quad (7)$$

In this work width of the beam is not constant but function of  $x$  as it is previously given in Equation (1), then it turns Equation (7) into

$$I(x) = \frac{1}{12} H^3 B(x). \quad (8)$$

### 3 Small deflection

The lup-like cantilever beam is tipped in the center of the head in  $z$  direction (perpendicular to the drawing plane in Figure 1 or along the drawing plane in Figure 2) so it deflects. For linear analysis of small deflection, the curvature  $\kappa$  of the deflected beam is approximated as [11]

$$\kappa = \frac{d^2 z}{dx^2}. \quad (9)$$

There is relation between curvature  $\kappa$ , elastic modulus  $E$ , bending moment along  $x$  axis  $\tau(x)$ , and area of moment inertia  $I$  [12]

$$\kappa = \frac{\tau(x)}{EI}. \quad (10)$$

If a force  $F$  is applied to the center of cantilever head as illustrated in Figure 2 with fixed end at  $x = 0$  and free end at  $x = L$ , then the bending moment would be

$$\tau(x) = \left[ x - \left( L - \frac{1}{2} B_t \right) \right] F. \quad (11)$$

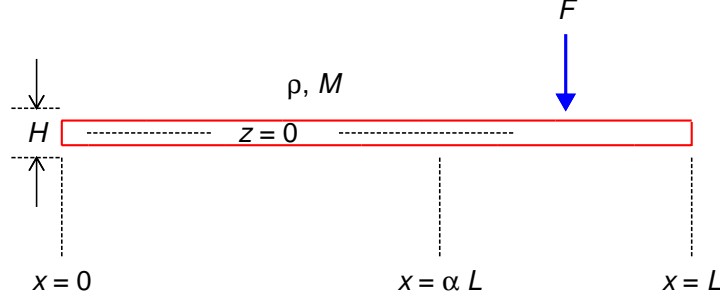


Figure 2: A force  $F$  is applied to center of the head of lup-like cantilever beam.

Substitute Equations (9) and (11) into Equation (10) will produce a second order differential equation

$$\frac{d^2 z}{dx^2} = \frac{F}{EI} \left[ x - \left( L - \frac{1}{2} B_t \right) \right], \quad (12)$$

whose solution is

$$z(x) = \frac{F}{EI} \left[ \frac{1}{6} x^3 - \frac{1}{2} \left( L - \frac{1}{2} B_t \right) x^2 \right] \quad (13)$$

which is a little bit different than for uniform rectangular cross section cantilever beam, which is tipped in the free end of the beam [13]. Equation (13) also assumes that area of moment inertia is constant.

Different result will be obtained if Equation (8) is substituted first into Equation (10) before solving the second order differential equation. Then following second order differential equation will be produced

$$\frac{d^2 z}{dx^2} = \frac{12F}{EH^3} \left[ x - \left( L - \frac{1}{2} B_t \right) \right] \times \begin{cases} B_0^{-1}, & 0 \leq x < \alpha L, \\ \left( 2\sqrt{\frac{1}{4} B_t^2 - \left[ x - \left( \alpha L + \frac{1}{2} \sqrt{B_t^2 - B_0^2} \right) \right]^2} \right)^{-1}, & \alpha L \leq x \leq L. \end{cases} \quad (14)$$

For  $0 \leq x < \alpha L$  the solution of Equation (14) is similar to Equation (13), which is

$$z(x) = \frac{12F}{EH^3 B_0} \left[ \frac{1}{6} x^3 - \frac{1}{2} \left( L - \frac{1}{2} B_t \right) x^2 \right]. \quad (15)$$

And for  $\alpha L \leq x \leq L$ , following constants

$$c_1 = \frac{6F}{EH^3}, \quad (16)$$

$$c_2 = L - \frac{1}{2}B_t = \alpha L + \frac{1}{2}\sqrt{B_t^2 - B_0^2}, \quad (17)$$

$$c_3 = \frac{1}{2}B_t, \quad (18)$$

and also functions and other constants

$$\cos \theta(x) = \frac{x - c_2}{c_3}, \quad (19)$$

$$\sin \theta(x) = \frac{\sqrt{c_3^2 - (x - c_2)^2}}{c_3}, \quad (20)$$

$$\cos \theta_\alpha = \frac{\sqrt{B_t^2 - B_0^2}}{B_t}, \quad (21)$$

$$\sin \theta_\alpha = \frac{B_0}{B_t}. \quad (22)$$

are defined. Constants and functions in Equation (16) - (22) will simplify Equation (14) for  $\alpha L \leq x \leq L$  to

$$\frac{d^2 z}{dx^2} = \frac{c_1(x - c_2)}{\sqrt{c_3^2 - (x - c_2)^2}} \quad (23)$$

First integration from  $x = \alpha L$  to  $x$  will turn Equation (23) into

$$\left. \frac{dz}{dx} - \frac{dz}{dx} \right|_{x=\alpha L} = c_1 \sqrt{c_3^2 - (x - c_2)^2} - \frac{1}{2}c_1 B_0. \quad (24)$$

Further integration within the same range will lead to

$$z(x) - z(\alpha L) = \frac{c_1 c_3^2}{2} \left[ \sin \theta(x) \cos \theta(x) - \frac{B_0 \sqrt{B_t^2 - B_0^2}}{B_t^2} \right] \quad (25)$$

$$- \theta(x) + \sin^{-1} \left( \frac{B_0}{B_t} \right) + \left[ \left. \frac{dz}{dx} \right|_{x=\alpha L} - \frac{c_1 B_0}{2} \right] (x - \alpha L) \quad (26)$$

Values of  $z(\alpha L)$  and its derivative  $[dz/dx](\alpha L)$  are obtained from Equation (15), which are

$$z(\alpha L) = \frac{12F}{EH^3 B_0} \left[ \frac{1}{2} \alpha^2 L^3 \left( \frac{\alpha}{3} - 1 \right) + \frac{1}{4} \alpha^2 L^2 B_t \right], \quad (27)$$

$$\left. \frac{dz}{dx} \right|_{x=\alpha L} = \frac{12F}{EH^3 B_0} \left[ \alpha L^2 \left( \frac{\alpha}{2} - 1 \right) + \frac{1}{2} \alpha L B_t \right]. \quad (28)$$

Using Equations (27) and (28) and by setting  $x = c_2$  deflection of the head of cantilever beam where the force  $F$  tips it can be found, which is

$$z(c_2) = \frac{12F}{EH^3B_0} \left[ \frac{1}{2}\alpha^2L^3 \left( \frac{\alpha}{3} - 1 \right) + \frac{1}{4}\alpha^2L^2B_t \right] - \frac{3FB_t^2}{4EH^3} \left[ \frac{B_0\sqrt{B_t^2 - B_0^2}}{B_t^2} + \frac{\pi}{2} - \sin^{-1} \left( \frac{B_0}{B_t} \right) \right] + \frac{1}{2}\sqrt{B_t^2 - B_0^2} \left\{ \frac{12F}{EH^3B_0} \left[ \alpha L^2 \left( \frac{\alpha}{2} - 1 \right) + \frac{1}{2}\alpha LB_t \right] - \frac{3FB_0}{EH^3} \right\}. \quad (29)$$

Identity from Equation (17) can simplify Equation (29) into

$$z(c_2) = \frac{F}{EH^3} \left\{ \frac{9}{2}B_0 \left[ (1 - \alpha)L + \frac{1}{2}B_t \right] + \frac{3}{4}B_t^2 \left[ \sin^{-1} \left( \frac{B_0}{B_t} \right) - \frac{\pi}{2} \right] + \frac{6\alpha LB_t}{B_0} \left[ (2 - \alpha)L - \frac{1}{2}B_t \right] - \frac{12L^3}{B_0} \left( \alpha - \alpha^2 + \frac{1}{3}\alpha^3 \right) \right\}. \quad (30)$$

For  $\alpha = 1$  and  $B_t = 0$  Equations (30) and (13) give the same result, which can be considered as proof for the first equation.

## 4 Spring constant and natural frequency

For small deflection in  $z$  direction of a cantilever beam with one fixed end and the other end is under influence of certain force  $F$ , the beam can be considered as a spring which has spring constant  $k$ . Then the beam obeys Hook's law

$$F = -kz. \quad (31)$$

Using result from Equation (30) spring constant of lup-like cantilever beam tipped in  $x = c_2$  can be found

$$k = - \left( \frac{EH^3B_0}{4L^3} \right) \left\{ \frac{9B_0^2}{8L^3} \left[ (1 - \alpha)L + \frac{1}{2}B_t \right] + \frac{3B_t^2B_0}{16L^3} \left[ \sin^{-1} \left( \frac{B_0}{B_t} \right) - \frac{\pi}{2} \right] + \frac{3\alpha B_t}{2L^2} \left[ (2 - \alpha)L - \frac{1}{2}B_t \right] - 3 \left( \alpha - \alpha^2 + \frac{1}{3}\alpha^3 \right) \right\}^{-1}. \quad (32)$$

The term in first ( ) is the spring constant for cantilever beam with constant rectangular cross section [14]. Or alternatively, Equation (32) can be written in form of

$$k = k_{\square}\beta, \quad (33)$$

where  $k_{\square}$  is spring constant for cantilever beam with constant rectangular cross section and  $\beta$  is correction factor for other form due to geometry difference

$$k_{\square} = \frac{EH^3B_0}{4L^3}, \quad (34)$$

$$\beta = - \left\{ \frac{9B_0^2}{8L^3} \left[ (1-\alpha)L + \frac{1}{2}B_t \right] + \frac{3B_t^2B_0}{16L^3} \left[ \sin^{-1} \left( \frac{B_0}{B_t} \right) - \frac{\pi}{2} \right] + \frac{3\alpha B_t}{2L^2} \left[ (2-\alpha)L - \frac{1}{2}B_t \right] - 3 \left( \alpha - \alpha^2 + \frac{1}{3}\alpha^3 \right) \right\}^{-1}. \quad (35)$$

This factor can also put in the frequency instead in the spring constant [8]. From this spring-mass system, where not all mass of the cantilever beam contributes to the oscillation (only effective mass  $m^*$  instead of the whole mass  $m$ ), the natural frequency can be found

$$\omega = \sqrt{\frac{k}{m^*}} \quad (36)$$

or explicitly

$$\omega = \sqrt{\frac{EH^3B_0\beta}{4L^3m^*}}. \quad (37)$$

## 5 Effective mass of the vibration

Equation of motion of a uniform beam, by neglecting shear deformation and rotary inertia, will lead to frequency equation [15]

$$1 + \cos \eta_n L + \cosh \eta_n L = 0, \quad n = 1, 2, \dots, \quad (38)$$

whose solutions are related to the natural frequency of the beam vibration (with constant rectangular cross section)

$$\omega_n = \eta_n^2 \sqrt{\frac{EIL}{m}} = \eta_n^2 \sqrt{\frac{EH^3BL}{12m}}. \quad (39)$$

For the lowest vibration frequency ( $n = 1$ ) solution of Equation (38) is about  $1.875/L$ . Then using Equations (37) and (39) and the solution of frequency equation

$$\begin{aligned}
\omega^2 &= \omega_1^2 \\
\frac{EH^3 B_0 \beta}{4L^3 m^*} &= \left( \frac{1.875}{L} \right)^4 \frac{EH^3 BL}{12m} \\
\frac{\beta}{m^*} &= \frac{1.875^4}{3m} \\
m^* &\approx 0.243\beta m.
\end{aligned} \tag{40}$$

For common cantiler beam with constant rectangular cross section value of  $\beta$  is 1 [10, 14]. It can be said that  $\beta$  shows the contribution of mass from the circular head of the cantiler beam.

## 6 Influence of $B_0$ , $B_t$ , and $L$

With a certain beam length  $L$  four values of  $B_0$ , which are 1, 2, 3, and 4, are used to plot  $\beta$  against  $B_t$  as illustrated in Figure 3.

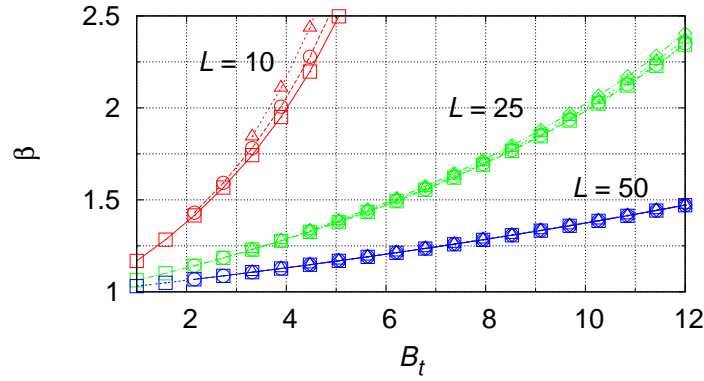


Figure 3: Plot of  $\beta$  as function of  $B_t$  for  $B_0$ : 1 ( $\square$ ), 2 ( $\circ$ ), 3 ( $\triangle$ ), and 4 ( $\diamond$ ).

It can be seen from Figure 3 that  $B_0$  does not play a significant role. All curves with different values of  $B_0$  seem to coincide. But values of  $B_t$  does change value of  $\beta$  significantly but still in the same order. The curves are also grouped with the same value of  $L$ , which means it has also a strong influence to  $\beta$ . Higher value of  $L$  will coincide different  $B_0$  better than lower one for the same range of  $B_t$ .

## 7 Conclusion

Derivation of mass, area of moment inertia, spring constant, and effective mass for lup-like cantiler beam are already presented in this work. A parameter  $\beta$  is



also defined, which relates effective mass for constant rectangular cross section cantilever beam to the discussed lup-like cantilever beam. Width of the beam  $B_0$  does not change significantly value of  $\beta$  but width of the head  $B_t$  and beam length  $L$  do, which has been showed graphically.

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